SPREADING OF A VISCOUS DROP ON IMPACT

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The problem of the inertialess spreading of a drop due to impact compression was solved by Reynolds on the assumption of constancy of the coefficient of viscosity and was used in [1] to analyze the sensitivity of liquid explosives. It is of interest to take into account inertia forces and the variation of the coefficient of viscosity μ with temperature since the heating of the liquid and the deceleration of the striker depend on these factors. The outcome of the solution for steady-state conditions is also discussed.

We will assume that the radius R of the base of the striker is the same as the initial radius of the thin cylindrical layer of viscous substance of thickness δ_0. The mass of the striker is m and its initial velocity is V_0. After simplifications which depend on the smallness of the ratio δ_0/R, the hydrodynamic equations can be written thus:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\mu\frac{\partial u}{\partial z}, \quad \frac{\partial p}{\partial z} = 0,$$

$$\frac{1}{r}\frac{\partial ur}{\partial r} + \frac{\partial v}{\partial z} = 0, \qquad u(r, 0, t) = 0, \qquad u(r, \delta, t) = 0,$$

$$v(r, 0, t) = 0, \; v(r, \delta, t) = w, \; p(R, t) = 0. \tag{1}$$

We will assume that the liquid does not conduct heat. Then the dissipated energy in a Lagrangian particle with coordinates r_0 and z_0 is conserved

$$\left(\frac{dT}{dt}\right)_{r_0,z_0} = \frac{\mu}{\rho_0 c_p}\left(\frac{\partial u}{\partial z}\right)^2_{r,t}, \quad \left(\frac{\partial r}{\partial t}\right)_{r_0,z_0} = u, \quad \left(\frac{\partial z}{\partial t}\right)_{r_0,z_0} = v,$$

$$T(r_0, z_0, 0) = 0, \qquad r(r_0, z_0, 0) = r_0, \qquad z(r_0, z_0, 0) = z_0. \tag{2}$$

If $R/c \ll \tau_1$ (c is the velocity of sound in the striker, $\tau_1 = \delta_0/v_0$ is the characteristic time of impact), the deceleration of the striker can be written as:

$$m\frac{dw}{dt} = 2\pi\int_0^R pr\,dr, \qquad \delta = \delta_0 + \int_0^t w\,dt, \qquad w(0) = -v_0. \tag{3}$$

We solve Eq. (1) by the method of moments. We take u in the form of a series

$$u = z(\delta - z)\,[f_0(r, \delta) + zf_1(r, \delta) + \ldots]$$

which satisfies the conditions of adhesion of the liquid at z = 0 and z = δ. Multiplying the equation of motion (1) by z^n (n = 0, 1, 2, ...) and integrating it over the thickness of the layer from 0 to δ we obtain a system of differential equations which is equivalent to (1) at the limit $n \to \infty$. Integrating the continuity equation with respect to z and confining ourselves to the zero approximation in Eq. (4) we find

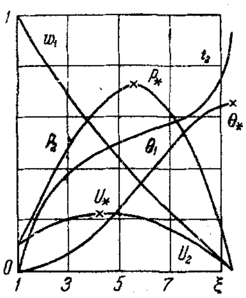

Fig. 1

$$v = -\frac{1}{r}\frac{\partial}{\partial r}r\int_0^z u\,dz, \qquad \text{i.e.} \quad u = \frac{3wrz(z-\delta)}{\delta^3},$$

$$v = \frac{wz^2(3\delta - 2z)}{\delta^3}. \tag{5}$$

Using Eqs. (2) and (5) and converting from t to δ, we establish a relationship between the moving and Lagrangian coordinates and determine the temperature distribution

$$\frac{dr}{u} = \frac{dz}{v} = \frac{d\delta}{w}, \qquad \frac{d\delta}{dt} = w, \qquad r = r_0\left[\frac{\delta_0(\delta + e)}{\delta(\delta_0 + e)}\right]^{3/4},$$

$$e = \frac{z_0\delta_0(\delta_0 - z_0)}{(0.5\delta_0 - z_0)^2},$$

$$z = \frac{\delta}{2}\left[1 \pm \sqrt{\delta/(\delta + e)}\right] \qquad \begin{pmatrix} + \text{ for } z > 0.5\delta \\ - \text{ for } z < 0.5\delta \end{pmatrix},$$

$$T = \frac{9}{\rho_0 c_p}\int_{\delta_0}^{\delta}\frac{\mu w r^2(\delta - 2z)^2}{\delta^6}\,d\delta, \qquad T_* = \frac{9\mu_0 R^2}{\rho_0 c_p}\int_{\delta_0}^{\delta}\frac{w}{\delta^4}\,d\delta. \tag{6}$$

It follows from (6) that the maximum temperature T_* is attained at the points r = R, z = 0, and r = R, z = δ. Using Eqs. (4) and averaging Eq. (1) over z, we find

$$\frac{\partial p}{\partial r} = \frac{6w\mu_0 r}{\delta^3}(\mu_1 + \psi), \quad \psi = \frac{\rho_0\delta^2}{12\mu_0}\frac{dw}{d\delta} - \frac{0.15\rho_0\delta w}{\mu_0},$$

$$\mu = \mu_0\mu_1[p, T(r, 0, t)]. \tag{7}$$

Here T(r, 0, t) is expressed from (6) on the condition z = 0 and, hence, $r = r_0$. Integration of Eq. (7) for arbitrary variation of the viscosity with p and T requires numerical calculation even within the

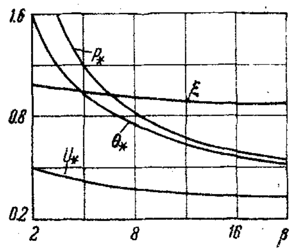

Fig. 2

framework of the assumptions made. However, in special cases the solution is obtained in quadratures

$$\int_0^p \frac{dx}{\mu_1(x) + \psi} = \frac{3w\mu_0}{\delta^3}(r^2 - R^2), \qquad \mu_1 = \mu_1(p),$$

$$\int_0^{y_1}\frac{\delta^3\,dx}{3w\mu_0[A(x/\theta)^\omega + \psi] - x\delta^3} = 2\ln\frac{r}{R},$$

$$y_1 = \frac{p}{r^2}, \qquad \omega = \frac{n}{n+1},$$

$$A = \left(\frac{\rho_0 c_p T_0 \delta_0^3}{9(n+1)\mu_0 v_0}\right)^\omega, \qquad \theta = \frac{T_*\rho_0 C_p\delta_0^3}{9\mu_0 R^2 v_0}, \qquad \mu_1 = \left(\frac{p}{T}\right)^n,$$

$$p = \frac{3w\mu_0}{\delta^3}\left[(r^2 - R^2)\psi + (n+1)A\theta^{-\omega}\left(r^{\frac{2}{n+1}} - R^{\frac{2}{n+1}}\right)\right],$$

$$\mu_1 = \left(\frac{T_0}{T}\right)^n. \tag{8}$$

Using Eq. (8) we obtain the equation of deceleration of the striker, which in dimensionless form for the case $\mu_1 = (T_0/T)^n$, for instance, is written thus:

$$(1+\lambda\beta\xi)\frac{d}{d\xi}\xi^{-2}\frac{d\theta}{d\xi}+1.8\lambda\beta\xi^{-2}\frac{d\theta}{d\xi}+\lambda\varkappa\xi\theta^{-\omega}=0,$$

$$\theta=\int_0^{\xi}\xi^2 w_1\,d\xi,$$

$$\theta(1)=0,\quad (d\theta/d\xi)_1=1,\quad w=-w_1v_0,$$

$$\xi=\delta_0/\delta,\quad \beta=0.15\rho_0v_0\delta_0/\mu_0,$$

$$\lambda=5\pi\mu_0R^4/6mv_0\delta_0^2,$$

$$\varkappa=3.6\,(n+1)\,AR^{-2\omega}/(n+2). \qquad (9)$$

This equation is easily integrated numerically. If $\beta\to 0$, then Eqs. (9) allows a similarity group and on substitution of the variables $\nu=\theta\xi^{\varepsilon}$, $z=d\nu/d\ln\xi$, $\varepsilon=-5(n+1)/(2n+1)$ reduces to an equation of the first order. If $n=0$, i.e., $\mu=$ const, then Eq. (9) is integrated in finite form

$$w_1=\left(\frac{y_0}{y}\right)^{1.8}+\frac{1}{\lambda\beta^2}\left[1-\left(\frac{y_0}{y}\right)^{1.8}\right]-\frac{9y}{14\lambda\beta^3}\left[1-\left(\frac{y_0}{y}\right)^{2.8}\right],$$

$$t_1=\int_1^{\xi}w_1^{-1}\xi^{-2}\,d\xi,$$

$$y=1+\lambda\beta\xi,\quad y_0=1+\lambda\beta,\quad t_1=t/\tau_1. \qquad (10)$$

It follows from (10) that at large ξ we have $w_1\approx 1/\lambda\beta^2-9\xi/14\beta$. Hence, the striker stops when the layer has a finite thickness $\xi=\xi_*$.

Close to $\xi=\xi_*$ we have $w_1=18\lambda\xi_*(\xi_*-\xi)/(1+\lambda\beta\xi_*)$ and, hence, when $\xi\to\xi_*$, the value of t_1 increases as $\ln[1/(\xi_*-\xi)]$.

At low Reynolds numbers the inertia forces become smaller than the viscosity forces and relationships (10) give the law of motion of the striker in the problem of [1]

$$w_1=1+0.9\lambda\,(1-\xi^2),$$

$$0.9\lambda t_1=\frac{1}{a^2}\left[1-\frac{1}{\xi}+\frac{1}{2a}\ln\frac{(a+\xi)(a-1)}{(a-\xi)(a+1)}\right],$$

$$a^2=\frac{1+0.9\lambda}{0.9\lambda}. \qquad (11)$$

It follows from (10) that the velocity of the striker decreases monotonically with time. However, the radial velocity of flow at the point $z_1=0.5\delta$, $r=R$ at instant ξ_1 and pressure at point $r=0$ for $\xi=\xi_2$ attain maxima $u_1(\xi_1)$ and $p_1(\xi_2)$, which are determined from the relationships

$$u_1=\frac{4\delta_0u(R,z_1,\xi)}{3v_0R}=\xi w_1,$$

$$p_1=\frac{\delta_0^3p(0,\xi)}{3\mu_0v_0R^2}=w_1\xi^2\frac{\beta w_1+\xi}{1+\lambda\beta\xi}, \qquad (12)$$

$$w_0(1-0.8\lambda\beta\xi_1)=1.8\mu\xi_1^2,$$

$$2\beta(1-0.8\lambda\beta\xi_2)w_2^2+3\xi_2(1-0.8\lambda\beta\xi_2)w_2=1.8\lambda_2^3,$$

$$w_0=w_1(\xi_1),\qquad w_2=w_1(\xi_2). \qquad (13)$$

When $\beta\to 0$ we find from Eqs. (11) and (13) that $\xi_1=a^1/_3{}^{0.5}$, $\xi_2=a\,{}^3/_5{}^{0.5}$. Figure 1 gives the results of calculations of the dimensionless pressure $p_2=0.01p_1$, the temperature $\theta_1=0.01\,\theta$, the time $t_2=0.5t_1$, and the velocities of the liquid $u_2=0.1u_1$ and striker w_1 in relation to ξ for $\beta=10$ and $\lambda=0.01$.

In Fig. 2 these quantities at the maximum points are plotted as functions of β: $u_*=u_2(\xi_1)$, $p_*=p_2(\xi_1)$, $\theta_1=\theta(\xi_1)$ and $\xi_0=0.1\xi_1$.

The graphs show that the maximum velocity of outward flow is several times greater than the initial velocity and the maximum pressure (at small β) is even two orders greater. If the energy of elastic strain of the striker is greater than $mv_0^2/2$, the compressibility of the striker must be taken into account. Since the compression time t_1 increases logarithmically, the characteristic time is of the order of unity.

We can consider the problem of the spread of a drop between two infinite plates. In this case the variable radius of the drop is expressed from the law of conservation of mass $R^2(t)\,\delta=R_0^2\,\delta_0$ and the problem is solved in the same way as above. However, such a procedure is valid only in cases where the cumulative splash of the liquid at $z=\delta/2$ can be neglected.

We will now estimate the correctness of the hypothesis of a quasi-stationary parabolic velocity profile in the Reynolds problem ($\beta\to 0$) in the case of impact compression. The hypothesis of quasistationarity is valid if the characteristic time of formation of the viscous velocity profile $\tau_2\ll\tau_1$. If we consider the dimensions, then $\tau_2\sim\rho_0\delta_0^2/\mu_0$. However, the solution of the simplified problem of development of a viscous flow shows that τ_2 is an order less. This is important, since the solution depends exponentially on the time.

We will consider an instant t_0, close to the initial instant, but such that the pressure over the thickness of the layer manages to even out: $\tau_0>\delta_0/c$. We will assume the velocity of the striker to be constant and we will take the initial distribution of u in the form $u(r,z,0)=v_0r/2\delta_0$, which satisfies the continuity equation. In the equations of motion (1) we neglect the inertial terms, and retain the derivative $\partial u/\partial t$. The solution of Eqs. (17) shows that this is valid everywhere, except for the layer near the wall, where when $t\to 0$ the velocity gradients $\sim t^{-0.5}$ are of the same order as the acceleration [2]. However, physical sense demands that $t>t_0$. To get rid of the "movable boundary" $\delta(t)$, we convert to new variables $\eta=z/\delta$ and $\tau=(\xi-1)/\delta_0$. Then the system of equations, like (1), is written thus:

$$\rho_0v_0\frac{\partial u}{\partial\tau}+\rho_0v_0\delta\frac{\partial u}{\partial\eta}+\frac{\partial\varphi}{\partial r}=\mu_0\frac{\partial^2u}{\partial\eta^2},$$

$$\int_0^1 u\,d\eta=\frac{v_0r}{2}\left(\tau+\frac{1}{\delta_0}\right),$$

$$\varphi=p\delta^2,\quad u(r,\eta,0)=v_0r/2\delta_0,\quad u(r,0,\tau)=0,$$

$$u(r,1,\tau)=0,\quad \varphi(R,\tau)=0. \qquad (14)$$

We omit the term with δ in (14), since it is of the order $v\partial u/\partial y$ of the discarded inertia terms. Performing the Laplace transformation we obtain equations for the velocity images $u\doteqdot U$ and the pressure analog $\varphi\doteqdot P$.

$$\rho_0v_0sU-\frac{\rho_0v_0^2}{2\delta_0}r+\frac{\partial P}{\partial r}=\mu_0\frac{\partial^2U}{\partial\eta^2},$$

$$\int_0^1 U\,d\eta=\frac{v_0r}{2s}\left(\frac{1}{S}+\frac{1}{\delta_0}\right),$$

$$U(r,0)=0,\quad U(r,1)=0,\quad P(R)=0. \qquad (15)$$

Integrating Eq. (15), we obtain

$$U=\frac{v_0r\,(s+\delta_0)\,x\,\mathrm{sh}\,x\eta\,\mathrm{sh}\,x(1-\eta)}{\delta_0s^2\,(x-\mathrm{th}\,x)\,\mathrm{ch}\,x},$$

$$P=\frac{\rho_0v_0^2\,(R^2-r^2)}{4\delta_0}\left[\frac{(s+\delta_0)\,x}{s\,(x-\mathrm{th}\,x)}-1\right],$$

$$x=0.5\left(\frac{\rho_0v_0s}{\mu_0}\right)^{1/2}. \qquad (16)$$

It is easy to find the asymptotic form of Eq. (16) when $s\to\infty$

$$U=\frac{v_0r}{2\delta_0s}\,[1+1/x-$$

$$-\exp(-2x\eta)-\exp[-2x(1-\eta)]+\ldots],$$

$$P=\frac{\rho_0v_0^2}{4}\left[\frac{1}{s}+\frac{1}{x\delta_0}\right](R^2-r^2),\qquad \text{for}\quad t\to 0,$$

$$u=\frac{v_0r_0}{2\delta_0}\left[1+4\left(\frac{\mu_0\tau}{\pi\rho_0v_0}\right)^{1/2}-\mathrm{Erf}\,\frac{\eta}{2}\left(\frac{\rho_0v_0}{\mu_0\tau}\right)^{1/2}-\right.$$

$$- \operatorname{Erf} \frac{1-\eta}{2} \left(\frac{\sigma_0 v_0}{\mu_0 \tau}\right)^{1/2} + \ldots \Big],$$

$$p = \frac{\rho_0 v_0^2}{4\delta^2} (R^2 - r^2) \left[\frac{2}{\delta_0} \left(\frac{\mu_0}{\pi \rho_0 v_0 \tau}\right)^{1/2} \exp(-\tau) + 1 + \ldots\right]. \quad (17)$$

It follows from (17) that p increases infinitely when $t \to 0$. This is due to the fact that instantaneous stoppage of the wall layer of liquid requires infinite friction forces $\sim t^{-0.5}$, counterbalanced by inertial forces and the pressure, which is assumed constant over the thickness of the layer. The formulas for the originals for $t > 0$ can be obtained from Eqs. (16) by expanding U and φ in power series and using the residue theorem [3]. Special points in the plane s lie on the left of the imaginary axis. At point $s = 0$ there is a pole of the second order. Simple poles are situated at the points $s_k = -4\mu\lambda_k^2/\rho_0 v_0$, $\operatorname{tg} \alpha_k = \lambda_k$ ($\lambda_1 = 4.493$) and, hence,

$$u = \frac{3 v_0 r \eta (1-\eta)}{\delta} +$$

$$+ \sum_{k=1}^{\infty} \frac{2 v_0 r (s_k + \delta_0) \operatorname{sh} \eta x_k \operatorname{sh} (1-\eta) x_k}{\delta_0 s_k x_k \operatorname{sh} x_k} \exp\left(-\frac{4\mu\lambda_k^2 \tau}{\rho_0 v_0}\right),$$

$$p = \frac{3 v_0 \mu_0 (R^2 - r^2)}{\delta^3} \times$$

$$\times \left[1 + 0.1 \frac{\rho_0 v_0 \delta}{\mu} + \frac{2}{3} \frac{\delta}{\delta_0} \sum \frac{\delta_0 + s_k}{s_k} \exp\left(-\frac{4\mu\lambda_k^2 \tau}{\rho_0 v_0}\right)\right]. \quad (18)$$

It is clear from (18) that the solution rapidly (with characteristic time $\tau_3 = \tau_2/4\lambda_1$) reaches the quasistationary regime.

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